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Testing Two Versions of Lattice Gauge Theory: Creutz Ratios in $U(1)_3$

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In our simplicial version of lattice gauge theory, we approximate Euclidean path integrals by tiling space-time with simplexes and by linearly interpolating the fields throughout each simplex from their values at the vertices. We compare this method with Wilson's lattice gauge theory for $U(1)$ in three dimensions. As a standard of comparison, we compute the exact values of Creutz ratios of Wilson loops in the continuum theory. Monte Carlo computations using the simplicial method give Creutz ratios within a few percent of the exact values for reasonably sized loop at $\beta = 1$, 2, and 10. Similar computations using Wilson's method give ratios that typically differ from the exact values by factors of 2 or more for $1 \leqslant \beta \leqslant 3.5$ and that have the wrong β dependence. The better accuracy of the simplicial method is due to its use of the action and domain of integration of the exact theory, unaltered apart from the granularity of the simplicial lattice. We also present data on the action density and the mass gap.

KEY WORDS: Lattice gauge theory; simplex; interpolation; Wilson loops; Creutz ratios; $U(1)$; Euclidean path integrals.

INTRODUCTION

In 1974 Wilson invented techniques that have since become known as lattice gauge theory.[1] Creutz[2-3] and others later showed Wilson's lattice gauge theory to be a practical way to study gauge theories nonperturbatively. However the action and domain of integration of Wilson's method differ from those of the exact theory unless the group elements are near the identity. Since the action and domain of integration control the sampling of

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fields in a Monte Carlo simulation, these differences may lead to errors at intermediate and strong coupling, where the group elements are not always near the identity.

In a theory with a running coupling constant, one may try to reduce these errors by working at weak coupling. But in $SU(3)$ the physical size of the lattice spacing shrinks with the coupling g like $\exp[-1/(2b_0 g^2)]$ with $b_0 = 11/(16\pi^2)$.[4] If one reduces the coupling, then one must compute on a larger lattice in order to encompass the same physical phenomenon. A reduction of g from 1.0 to 0.5 requires an increase of a 10^4 lattice to one of size $(10^{10})^4$, which is inconveniently large.

In an earlier paper,[5] we proposed and tested a method for approximating Euclidean path integrals at arbitrary coupling. In this method one tiles space-time with simplexes and linearly interpolates the fields throughout each simplex from their values at the vertices. The fields are defined throughout space-time. The method uses the action and domain of integration of the exact theory, unaltered apart from the granularity of the simplicial lattice. The simplicial method is similar to the finite-element method of C. M. Bender, G. S. Guralnik, and D.S. Sharp,[6] and differs from it by its use of linear interpolations in simplexes rather than nonlinear splines in cubes.

Here we describe an application of this simplicial method to $U(1)$ in three dimensions. By using a heat-bath Monte Carlo,[7] we computed the mean value of the action density, the (vanishing) mass gap, and the Creutz ratios of various quartets of Wilson loops. We derived an exact formula for the Creutz ratios $\chi(L, J)$ of $U(l)_3$ and used it as a standard to compare the accuracy of the simplicial method with that of Wilson's lattice gauge theory. We made this comparison at three values of the dimensionless inverse temperature $\beta = 1/(ag^2)$, where a is the lattice spacing. For loops that have a spatial extent La of between 4 and 7 lattice spacings and a temporal extent Ja of between 2 and 8 spacings, the simplicial method on a $16^2 \times 64$ lattice gave Creutz ratios $\chi(L, J)$ within 20% of the exact values for strong coupling $\beta = 1$, intermediate coupling $\beta = 2$, and weak coupling $\beta = 10$. For these loops and these βs, the average error of our Creutz ratios is 5.9%. In contrast, Wilson's lattice gauge theory, to judge from results reported by Bhanot and Creutz[8] and by Ambjørn, Hey, and Otto,[9] gives Creutz ratios that typically differ from the exact values by factors of 2 or more for $1 \leqslant \beta \leqslant 3.5$ and that have the wrong β dependence. But according to the weak-coupling expansion of Müller and Rühl,[10] Wilson's method on an infinite lattice should display acceptable errors at $\beta = 10$.

ACTIONS AND DOMAINS OF INTEGRATION

The basic variables of the Wilson method are the elements of the gauge group. These group elements, which are associated with the links of the lattice, may be parameterized in terms of the gauge fields A_n^α as $U_n = \exp(igaA_n^\alpha \lambda_\alpha)$. For $U(1)$ the elements of the gauge group are simply the phases $U_n = \exp(igaA_n)$ where A_n is the gauge field.

In Wilson's method, the group elements, one for each link, run independently over the manifold of the gauge group. The group manifold is compact and (except for Abelian groups) curved. The domain of integration is the direct product of copies of the group manifold, one for each link. In the exact continuum theory, the domain of integration is the product of copies of the real line, one for each gauge field at every space-time point. Even apart from the inevitable granularity of the lattice, the two domains of integration are very different. However, for weak coupling, the dimensionless angles $|gaA_n^\alpha|$ or $|gaA_n|$ stay small and the group elements remain close to the identity where the metric of the group manifold is nearly unity. Thus, for weak coupling, Wilson's domain of integration effectively resembles that of the exact theory. But for stronger coupling, the topology and curvature of the group manifold become important.

Apart from a constant, the Wilson action is proportional to the sum, over the elementary squares of the lattice, of the real part of the trace of the path-ordered product of the (oriented) group elements of the links around the square. To lowest order in the dimensionless angles $|gaA_n^\alpha|$ or $|gaA_n|$, the Wilson action is equal to that of the continuum theory, apart from the granularity of the lattice. Thus for weak coupling, Wilson's action is close to that of the exact theory. But at stronger coupling the higher-order terms in Wilson's action become important.

THE SIMPLICIAL METHOD

In the simplicial method, space-time is filled with a cubic lattice each cube of which is tiled with six (tetrahedral) simplexes. Equivalent simplexes in different cubes are oriented the same way. Each space-time point x lying in a simplex with vertices v_i can be uniquely expressed in the form

$$x = \sum_{i=0}^{3} \rho_i v_i \tag{1}$$

in which the nonnegative weights ρ_i sum to 1. We use this same formula to linearly interpolate the field $A_n(x)$ at the point x from its values $A(n, v_i)$ at the vertices v_i

$$A_n(x) = \sum_{i=0}^{3} \rho_i A(n, v_i) \tag{2}$$

Since the ρs sum to unity, ρ_0 is 1 minus the sum of the components of the three-vector $\rho = (\rho_1, \rho_2, \rho_3)$. One may obtain this three-vector by inverting the 3×3 matrix M defined as $M_{ji} = (v_i)_j - (v_0)_j$ and by forming the matrix product $\rho = M^{-1}(x - v_0)$. The resulting formula (2) for the gauge field depends on the simplex the point x is in. For the simplex whose vertices are $v_0 = (i, j, k)a$, $v_1 = (i+1, j, k)a$, $v_2 = (i+1, j+1, k)a$, and $v_3 = (i+1, j+1, k+1)a$, the interpolated gauge field at the point $x = (x, y, t)$ is

$$\begin{aligned} A_n(x) = {} & [(i+1)a - x]\, A(n, i, j, k) \\ & + [(j-i)a + x - y]\, A(n, i+1, j, k) \\ & + [(k-j)a + y - t]\, A(n, i+1, j+1, k) \\ & + (t - ka)\, A(n, i+1, j+1, k+1) \end{aligned} \tag{3}$$

Similar formulas obtain for the other five generic simplexes.

The field strength of the interpolated field $A_n(x)$ is $F_{mn}(x) = \partial_n A_m(x) - \partial_m A_n(x)$. The field A is continuous, but its derivatives have step-function discontinuities on the boundaries between simplexes. The field strength F inherits these integrable singularities. A key feature of the present method is that the interpolated fields are defined throughout space-time. It is therefore possible to use the action of the continuum theory unaltered apart from the granularity of the simplicial lattice. Thus we define the Euclidean action of the interpolated field $A_n(x)$ to be the integral over space-time of the sum of the squares of the interpolated field strengths

$$S(A) = \int d^3x \tfrac{1}{4} F_{mn}(x)^2 \tag{4}$$

Because the interpolation is linear in the field variables A, the Jacobian $\det[\partial A_n(x)/\partial A(m, i, j, k)]$ cancels in ratios of path integrals. By restricting space-time to a finite volume, one thus approximates the vacuum-expected value of a Euclidean time-ordered operator $\mathbf{Q}(\mathbf{A})$ by a multiple integral over the $A(n, i, j, k)$s

$$\langle 0|\, T\mathbf{Q}(\mathbf{A})\, |0\rangle \approx \int \prod dA(n, i, j, k) e^{-S(A)}\, Q(A) \Big/ \int \prod dA(n, i, j, k) e^{-S(A)} \tag{5}$$

where $Q(A)$ is obtained from $\mathbf{Q}(\mathbf{A})$ by replacing the operator $\mathbf{A}_n(x)$ with the interpolated field $A_n(x)$.

We used the temporal gauge so as to have only two gauge fields $A_n(x)$, $n = 1$ and 2, but this is not an essential feature of the simplicial method. As in Wilson's method, it is not necessary to fix the gauge. We used three lattices: 16^3, $16^2 \times 32$, and $16^2 \times 64$. For the small lattice with 4,096 vertices, the action is a quartic polynomial in 8,192 variables, the $A(m, i, j, k)$. For the medium and large lattices with 8,192 and 16,384 vertices, the action involves 16,384 and 32,768 As, respectively. However, because each field variable $A(m, i, j, k)$ influences the action in only 24 simplexes, it is coupled to only 30 of the 8,192, 16,384, or 32,768 variables.

We used the fact that the action is quadratic to write a heat-bath algorithm[7] that only requires knowledge of the first and second derivatives of the action with respect to each field variable. The algorithm constructs the parabola that describes the dependence of the action on each A, goes to the minimum of the parabola, and then adds noise at the inverse temperature β. In terms of the dimensionless scaled field variable $\alpha(n, i, j, k) \equiv agA(n, i, j, k)$, the formula for $d\alpha$ is

$$d\alpha = -S'/S'' + x(S'')^{-1/2} \tag{6}$$

in which S' and S'' are the first and second derivatives of the action with respect to the field variable α, and x is a random number normally distributed on $(-\infty, \infty)$. We used URAND[11] and the polar method of Box, Muller, and Marsaglia[12] to write a fast subroutine that generates a few thousand xs per call. The update $d\alpha$ of each field variable is as big as required, and thermalization is quick. We used MACSYMA[3] to calculate the derivatives S' and S'' and to express them in FORTRAN. For the $16^2 \times 64$ lattice, each sweep took 17 s on a Ridge 32C computer with 4MB of RAM.

THE ACTION DENSITY

The most accessible of the physical quantities we measured is the mean value of the action per cube, $\langle S \rangle_c$. The corresponding quantity in Wilson's lattice gauge theory is the mean value of the action per plaquette, $\langle S \rangle_\Box$, or of the internal energy per plaquette, $\langle P \rangle = \langle S \rangle_\Box / \beta$. In three dimensions, $\langle S \rangle_c = 3\langle S \rangle_\Box = 3\beta\langle P \rangle$.

The action is a quadratic form which may be diagonalized. Thus the mean value of the action in a single cube can be computed exactly: it is

[3] MACSYMA is a symbol manipulator developed by MIT and marketed by Symbolics, Inc. (617-577-7350).

just $\langle S \rangle_c = (N - N_0)/(2N_c)$ where N is the number of αs, N_0 is the number of zero modes, and $N_c = N/2$ is the number of cubes in the lattice. Equivalently, $\langle S \rangle_c = 1 - (N_0/N)$. In the temporal gauge, the zero modes of the simplicial lattice are due to the invariance of the action of the continuum theory under time-independent gauge transformations. We determined the number of zero modes by using MACSYMA and MATLAB[13] to form and diagonalize the quadratic part of the magnetic action for a spatial slab of an $N_s^2 \times N_t$ lattice. We found $N_0 = 5 + 3(N_s - 2)$.

Thus, for the 16^3 lattice with $N = 8192$ there are $N_0 = 47$ zero modes, and the mean value of the action per cube is $\langle S \rangle_c = 0.9943$. A run from a cold start, i.e., all αs zero, at $\beta = 2$ gave $\langle S \rangle_c = 0.9933$ after 8000 sweeps of which the last 7000 were averaged.

For the $16^2 \times 32$ lattice with $N = 16{,}384$, there are $N_0 = 47$ zero modes, and the mean value of the action per cube is $\langle S \rangle_c = 0.9971$. A run from a cold start at $\beta = 10$ gave $\langle S \rangle_c = 0.9974$ after 2450 sweeps of which the last 1800 were averaged. A similar run at $\beta = 2$ gave $\langle S \rangle_c = 0.9977$ after 2785 sweeps of which the last 800 were averaged.

For the $16^2 \times 64$ lattice, $N = 32{,}768$, $N_0 = 47$, and $\langle S \rangle_c = 0.9986$. A run from a cold a cold start at $\beta = 10$ gave $\langle S \rangle_c = 0.9982$ after 2457 sweeps of which the last 750 were averaged. Similar runs gave $\langle S \rangle_c = 0.9981$ at $\beta = 2$ after 2560 sweeps, of which the last 600 were averaged, and $\langle S \rangle_c = 0.9990$ at $\beta = 1$ after 1650 sweeps, of which the last 550 were averaged.

Because the theory is quadratic, the mean value of the action per cube is independent of the coupling β. It is therefore pointless to search for hysteresis loops or phase transitions.

THE VANISHING MASS GAP

There is no mass gap in this theory. We verified that our Monte Carlo gave a vanishing mass gap by using it to compute the product of the mass gap E times the lattice spacing a from the formula

$$aE = \lim_{\kappa \to \infty} \ln \left[\frac{\langle \alpha(n, i, j, \kappa + 1 + k)\, \alpha(n, i, j, k) \rangle}{\langle \alpha(n, i, j, \kappa + k)\, \alpha(n, i, j, k) \rangle} \right] \tag{7}$$

a sum being understood over i, j, and k in both the numerator and the denominator. On the 16^3 lattice, a run of 8000 sweeps and 100 measurements, starting from fields thermalized with 16,000 sweeps at $\beta = 2$, gave for $\kappa = 0$ through 6 the values $aE = 0.0020$, 0.0011, 0.0008, 0.0005, 0.0004, 0.0002, and 0.00006. On the $16^2 \times 64$ lattice, a run of 200 sweeps and 100 measurements, starting from fields thermalized with 12,700 sweeps at $\beta = 1$, gave for $\kappa = 0$ through 6 the values $aE = 0.0204$, 0.0119, 0.00087, 0.0071, 0.0061, 0.0053, and 0.0049.

EXACT FORMULA FOR CREUTZ RATIOS

The Wilson loop (functional) is the mean value in the vacuum of a path-ordered exponential of a line integral along a loop of the Euclidean connection. In his original article on lattice gauge theory,[1] Wilson pointed out that evidence of confinement could be obtained from an area term in the logarithm of the Wilson loop. In 1979 it was recognized that Wilson loops vanish in more than two dimensions due to a string singularity in the line integral of the connection.[14] Later Creutz introduced the practice of measuring the ratio of products of Wilson loops in a way that separates the physically important area term from this singularity.[2] In order to have a standard by which to judge the simplicial interpolative method and to compare it to Wilson's method, we shall now calculate the Creutz ratios of Wilson loops exactly in the continuum theory.

In temporal-gauge $U(1)$, the Wilson loop is the mean value in the vacuum of the Euclidean time-ordered product

$$W(r, t) = \langle 0|\, T \exp\left[ig \int \mathbf{A}_n(x)\, dx_n \right] |0\rangle \tag{8}$$

in which the integral runs from 0 to r at time 0 and from r to 0 at time t. It will be useful to consider the generating functional $W[j] = \langle 0|\, T\exp[i \int \mathbf{A}_n(x)\, j_n(x)\, d^3x]\, |0\rangle$. On the one hand, the Wilson loop is the generating functional $W[j]$ for the current

$$j_n(x) = \delta_{n,2}\, \delta(x_1)\, \theta(x_2)\, \theta(r - x_2)[\delta(x_0) - \delta(x_0 - t)] \tag{9}$$

On the other hand, the generating functional is well-known to be given by the formula

$$W[j] = \exp\left[-(g^2/2) \int j_m(x)\, D_{mn}(x, y)\, j_n(y)\, d^3x\, d^3y \right] \tag{10}$$

where the temporal-gauge Euclidean two-point function is

$$D_{mn}(x, y) = (2\pi)^{-3} \int k^{-2}(\delta_{mn} + \kappa_0^{-2} k_m k_n) \exp[ik \cdot (x - y)] \tag{11}$$

in which $k = (k_0, k_1, k_2)$ and $k^2 = k \cdot k$. By substituting the current $j_n(x)$

into the formula for the generating functional $W[j]$, we find for the Wilson loop the expression

$$\ln[W(r, t)] = -(g^2/\pi^3)[U(r, t) + U(t, r)] \tag{12}$$

where $U(r, t)$ is the integral

$$U(r, t) = \int d^3k \sin(k_2 r/2)^2 \sin(k_0 t/2)^2/(k^2 k_2{}^2) \tag{13}$$

This integral is logarithmically divergent due to a string singularity[14] inherent in the current $j_n(x)$, and the Wilson loop $W(r, t)$ therefore vanishes. This divergence may be canceled by forming the Creutz ratio[2] of products of related Wilson loops

$$\chi(L, J) \equiv \chi(r/a, t/a) = -\ln\left[\frac{W(r, t)\, W(r-a, t-a)}{W(r-a, t)\, W(r, t-a)}\right] \tag{14}$$

The Creutz ratio is insensitive to terms in $\ln(W)$ that are independent of r and t or that depend linearly on r or t. Whenever the loops are dominated by an area law, $\chi(L, J)$ directly measures the string tension, $\chi(L, J) \approx \sigma(\beta)$. In Wilson's lattice gauge theory, this occurs when L and J are large or when the coupling is large.[3]

It is useful to write the Creutz ratio in the form

$$\begin{aligned}\chi(r/a, t/a) = (g^2/\pi^3)[&U(r, t) - U(r-a, t)\\ &+ U(t, a) - U(t-a, r)\\ &+ U(r-a, t-a) - U(r, t-a)\\ &+ U(t-a, r-a) - U(t, r-a)]\end{aligned} \tag{15}$$

because the difference of two Us with the same second argument is the difference of two convergent integrals

$$U(r, t) - U(r', t) = C(r', t) - C(r, t) \tag{16}$$

the integral $C(r, t)$ being

$$\begin{aligned}C(r, t) &= 2\pi \int_0^\infty dx \int_0^\infty dy [\sin(rx/2)/x]^2 \cos(ty)/(x^2 y^2)^{1/2}\\ &= 2\pi t \int_0^\infty dx [\sin(rx/2t)/x]^2 K_0(x)\end{aligned} \tag{17}$$

where K_0 is Macdonald's function. The second derivative of $C(r, t)$ with respect to r/t is a known integral. After integrating twice with respect to r/t, one finds

$$\begin{aligned} C(r, t) &= \pi^2 t/2 \int_0^{r/t} dx \ln[x + (1 + x^2)^{1/2}] \\ &= (\pi^2 t/2)\, F(r/t) \end{aligned} \tag{18}$$

in which the function $F(q)$ is defined by

$$F(q) = 1 - q - [q + (1 + q^2)^{1/2}]^{-1} + q \ln[q + (1 + q^2)^{1/2}] \tag{19}$$

Our final formula for the exact Creutz ratio in $U(1)_3$ is, therefore

$$\begin{aligned} \chi(L, J) = (2\pi\beta)^{-1}\{&(J-1)\, F[L/(J-1)] - JF(L/J) \\ &+ (L-1)\, F[J/(L-1)] - LF(J/L) \\ &+ JF[(L-1)/J] - (J-1)\, F[(L-1)/(J-1)] \\ &+ LF[(J-1)/L] - (L-1)\, F[(J-1)/(L-1)]\} \end{aligned} \tag{20}$$

in which $L = r/a$ and $J = t/a$ are the extents of the largest loop of the ratio in units of the lattice spacing a. The exact χ is a symmetric function $\chi(L, J) = \chi(J, L)$ and depends upon β exclusively through the factor $1/\beta = ag^2$, as expected in a free theory. In the limit of large J, $\chi(L, J) \approx (2\pi\beta)^{-1} \ln[L/(L-1)]$ which is the difference $a[V(La) - V(La - a)]$ where $V(r)$ is the static potential of two charges g separated by the distance r: $V(r) = (g^2/2\pi) \ln(r)$.

MONTE CARLO CREUTZ RATIOS FROM THE SIMPLICIAL METHOD

To compute Creutz ratios on the 16^3 lattice at $\beta = 2$, we did 16,000 sweeps starting from thermalized fields and measuring Wilson loops every 8 sweeps. The Creutz ratios $\chi(L, 2)$ for $L = 4$ through 8 are within 4% of the exact ratios, and $\chi(3, 2)$ is off by only 7%. The $\chi(L, 3)$s are within 7 to 15% of the exact values for $L = 4$ through 8. The $\chi(L, 4)$s have errors of 9 to 27% for $L = 4$ through 8.

The accuracy of the χs increases with L, the number of vertices in the spatial direction, and decreases with J, the number in the temporal direction. In the temporal gauge, the loops run only in spatial directions; the missing temporal legs must be made up by gauge invariance, which is enforced only by the (unnormalized) projection operator $\exp(-HT)$ where

T is the temporal extent. The temporal extent of this lattice is $16a$, which is small compared to the inverse of the nearly vanishing mass gap. The rate at which the errors increase with J is smaller on the $16^2 \times 32$ and $16^2 \times 64$ lattices by factors of 2 and 4, respectively. These errors at large J are therefore a finite-duration effect due to our use of the temporal gauge and exacerbated by the absence of a mass gap. The errors due to the granularity of the simplicial lattice are small when L is at least 4.

To compute Creutz ratios on the $16^2 \times 32$ lattice at $\beta = 2$, we started with thermalized fields and did 3700 sweeps with measurements every 2 sweeps. For $L = 4$ through 8 and $J = 2$ and 3, the $\chi(L, J)$s are within 3% of the exact values, except for $\chi(4, 3)$ which is off by 7%. The $\chi(L, 4)$s have errors of from 3 to 16% for $L = 4$ through 8. To compute Creutz ratios on this $16^2 \times 32$ lattice at $\beta = 10$, we did 4800 sweeps with measurements every 3 sweeps starting with thermalized fields. The errors of these Creutz ratios are similar to those gotten at $\beta = 2$ on this lattice. Both sets of errors are about half as big as those of the 16^3 lattice.

Table I contains the Creutz ratios we obtained from runs at $\beta = 1$ on the $16^2 \times 64$ lattice. To compute these ratios, we made 12,000 sweeps with measurements every other sweep, starting from fields thermalized by 700 sweeps. For $L = r/a = 5$ through 7 and $J = t/a = 2$ through 6, these $\chi(L, J)$s are within 7% of the exact values. For $L = 3$ and 4 and $J = 2$ and 3, they have errors of 6% or less.

Table II contains the Creutz ratios we obtained from runs at $\beta = 2$ on the $16^2 \times 64$ lattice. To compute these ratios we did 2750 sweeps with measurements every third sweeps, starting from thermalized fields. For $L = 6$ through 8 and $J = 2$ through 8, these $\chi(L, J)$s are within 8% of the exact values. For $L = 4$ and 5 and $J = 2$ through 4, they have errors of less than 6%.

Table III contains the Creutz ratios we obtained from runs at $\beta = 10$ on the $16^2 \times 64$ lattice. To compute these ratios we did 1650 sweeps with measurements every third sweep, starting from thermalized fields. For $L = 5$ through 8 and $J = 2$ through 8, these $\chi(L, J)$s are within 7% of the exact values, with an average error of only 3.8%. For $L = 3$ and 4 and $J = 2$ through 4, they have errors running from 1 to 11%.

For all three values of β, the Creutz ratios $\chi(L, J)$ we got from runs on the $16^2 \times 64$ lattice are within 20% of the exact values for $L = r/a = 4$ through 7 and $J = t/a = 2$ through 8. The average error of these χs is 5.9%. For $L = 5$ through 7 and $J = 2$ through 6, the average error of the $\chi(L, J)$s is 3.5% at $\beta = 1$. For $L = 6$ through 8 and $J = 2$ through 8, the average error of the $\chi(L, J)$s is 4.0% at $\beta = 2$ and 3.9% at $\beta = 10$.

Table I. Creutz Ratios $\chi(r/a, t/a)$ at $\beta = 1$

Monte Carlo χs	Exact χs	% error
$\chi(3, 2) = 0.12635 \pm 0.00039$	0.12845	-1.63
$\chi(3, 3) = 0.08562 \pm 0.00063$	0.09157	-6.50
$\chi(3, 4) = 0.06870 \pm 0.00091$	0.07930	-13.38
$\chi(3, 5) = 0.05917 \pm 0.00120$	0.07378	-19.80
$\chi(3, 6) = 0.05327 \pm 0.00151$	0.07084	-24.80
$\chi(3, 7) = 0.04896 \pm 0.00181$	0.06910	-29.15
$\chi(3, 8) = 0.04577 \pm 0.00213$	0.06799	-32.68
$\chi(4, 2) = 0.12122 \pm 0.00066$	0.11979	1.19
$\chi(4, 3) = 0.07902 \pm 0.00108$	0.07930	-0.36
$\chi(4, 4) = 0.06196 \pm 0.00155$	0.06486	-4.48
$\chi(4, 5) = 0.05146 \pm 0.00207$	0.05802	-11.30
$\chi(4, 6) = 0.04702 \pm 0.00262$	0.05426	-13.34
$\chi(4, 7) = 0.04149 \pm 0.00317$	0.05198	-20.19
$\chi(4, 8) = 0.03993 \pm 0.00374$	0.05051	-20.95
$\chi(5, 2) = 0.11852 \pm 0.00100$	0.11611	2.08
$\chi(5, 3) = 0.07570 \pm 0.00167$	0.07378	2.61
$\chi(5, 4) = 0.05780 \pm 0.00242$	0.05802	-0.38
$\chi(5, 5) = 0.04714 \pm 0.00326$	0.05028	-6.24
$\chi(5, 6) = 0.04311 \pm 0.00418$	0.04590	-6.08
$\chi(5, 7) = 0.03699 \pm 0.00510$	0.04319	-14.35
$\chi(5, 8) = 0.03635 \pm 0.00612$	0.04141	-12.22
$\chi(6, 2) = 0.11749 \pm 0.00145$	0.11422	2.86
$\chi(6, 3) = 0.07418 \pm 0.00248$	0.07084	4.72
$\chi(6, 4) = 0.05470 \pm 0.00366$	0.05426	0.80
$\chi(6, 5) = 0.04404 \pm 0.00498$	0.04590	-4.05
$\chi(6, 6) = 0.04118 \pm 0.00648$	0.04107	0.27
$\chi(6, 7) = 0.03467 \pm 0.00807$	0.03802	-8.82
$\chi(6, 8) = 0.03380 \pm 0.00993$	0.03598	-6.70
$\chi(7, 2) = 0.11701 \pm 0.00206$	0.11312	3.44
$\chi(7, 3) = 0.07232 \pm 0.00363$	0.06910	4.67
$\chi(7, 4) = 0.05571 \pm 0.00549$	0.05198	7.16
$\chi(7, 5) = 0.04171 \pm 0.00764$	0.04319	-3.42
$\chi(7, 6) = 0.03675 \pm 0.01019$	0.03802	-3.34
$\chi(7, 7) = 0.03867 \pm 0.01309$	0.03471	11.39
$\chi(7, 8) = 0.02755 \pm 0.01655$	0.03247	-15.17
$\chi(8, 2) = 0.11589 \pm 0.00286$	0.11243	3.08
$\chi(8, 3) = 0.07050 \pm 0.00528$	0.06799	3.70
$\chi(8, 4) = 0.05892 \pm 0.00832$	0.05051	16.65
$\chi(8, 5) = 0.04385 \pm 0.01200$	0.04141	5.90
$\chi(8, 6) = 0.02800 \pm 0.01644$	0.03598	-22.18
$\chi(8, 7) = 0.04650 \pm 0.02208$	0.03247	43.20
$\chi(8, 8) = 0.02169 \pm 0.02869$	0.03007	-27.85

Table II. Creutz Ratios $\chi(r/a, t/a)$ at $\beta = 2$

Monte Carlo χs	Exact χs	% error
$\chi(3, 2) = 0.06324 \pm 0.00046$	0.06422	−1.53
$\chi(3, 3) = 0.04242 \pm 0.00074$	0.04579	−7.35
$\chi(3, 4) = 0.03411 \pm 0.00103$	0.03965	−13.98
$\chi(3, 5) = 0.02925 \pm 0.00135$	0.03689	−20.72
$\chi(3, 6) = 0.02681 \pm 0.00169$	0.03542	−24.32
$\chi(3, 7) = 0.02434 \pm 0.00204$	0.03455	−29.55
$\chi(3, 8) = 0.02256 \pm 0.00242$	0.03399	−33.64
$\chi(4, 2) = 0.06078 \pm 0.00075$	0.05989	1.48
$\chi(4, 3) = 0.03911 \pm 0.00121$	0.03965	−1.36
$\chi(4, 4) = 0.03054 \pm 0.00170$	0.03243	−5.85
$\chi(4, 5) = 0.02555 \pm 0.00223$	0.02901	−11.93
$\chi(4, 6) = 0.02385 \pm 0.00280$	0.02713	−12.10
$\chi(4, 7) = 0.02085 \pm 0.00338$	0.02599	−19.78
$\chi(4, 8) = 0.01922 \pm 0.00400$	0.02525	−23.91
$\chi(5, 2) = 0.05954 \pm 0.00109$	0.05805	2.55
$\chi(5, 3) = 0.03757 \pm 0.00176$	0.03689	1.84
$\chi(5, 4) = 0.02849 \pm 0.00245$	0.02901	−1.80
$\chi(5, 5) = 0.02318 \pm 0.00321$	0.02514	−7.80
$\chi(5, 6) = 0.02244 \pm 0.00404$	0.02295	−2.21
$\chi(5, 7) = 0.01860 \pm 0.00488$	0.02160	−13.89
$\chi(5, 8) = 0.01737 \pm 0.00577$	0.02071	−16.11
$\chi(6, 2) = 0.05868 \pm 0.00148$	0.05711	2.76
$\chi(6, 3) = 0.03675 \pm 0.00236$	0.03542	3.77
$\chi(6, 4) = 0.02763 \pm 0.00329$	0.02713	1.85
$\chi(6, 5) = 0.02151 \pm 0.00431$	0.02295	−6.27
$\chi(6, 6) = 0.02170 \pm 0.00544$	0.02053	5.69
$\chi(6, 7) = 0.01767 \pm 0.00657$	0.01901	−7.04
$\chi(6, 8) = 0.01672 \pm 0.00777$	0.01799	−7.05
$\chi(7, 2) = 0.05819 \pm 0.00190$	0.05656	2.89
$\chi(7, 3) = 0.03574 \pm 0.00303$	0.03455	3.44
$\chi(7, 4) = 0.02647 \pm 0.00421$	0.02599	1.84
$\chi(7, 5) = 0.02037 \pm 0.00554$	0.02160	−5.69
$\chi(7, 6) = 0.02034 \pm 0.00700$	0.01901	7.02
$\chi(7, 7) = 0.01680 \pm 0.00849$	0.01736	−3.23
$\chi(7, 8) = 0.01641 \pm 0.01007$	0.01624	1.08
$\chi(8, 2) = 0.05798 \pm 0.00237$	0.05621	3.13
$\chi(8, 3) = 0.03478 \pm 0.00379$	0.03399	2.32
$\chi(8, 4) = 0.02536 \pm 0.00528$	0.02525	0.42
$\chi(8, 5) = 0.01989 \pm 0.00694$	0.02071	−3.93
$\chi(8, 6) = 0.01935 \pm 0.00878$	0.01799	7.53
$\chi(8, 7) = 0.01569 \pm 0.01069$	0.01624	−3.39
$\chi(8, 8) = 0.01574 \pm 0.01277$	0.01503	4.70

Table III. Creutz Ratios $\chi(r/a, t/a)$ at $\beta = 10$

Monte Carlo χs	Exact χs	% error
$\chi(3, 2) = 0.01270 \pm 0.00012$	0.01284	-1.11
$\chi(3, 3) = 0.00866 \pm 0.00020$	0.00916	-5.44
$\chi(3, 4) = 0.00710 \pm 0.00029$	0.00793	-10.45
$\chi(3, 5) = 0.00628 \pm 0.00039$	0.00738	-14.92
$\chi(3, 6) = 0.00572 \pm 0.00049$	0.00708	-19.26
$\chi(3, 7) = 0.00532 \pm 0.00061$	0.00691	-23.01
$\chi(3, 8) = 0.00501 \pm 0.00072$	0.00680	-26.38
$\chi(4, 2) = 0.01214 \pm 0.00019$	0.01198	1.32
$\chi(4, 3) = 0.00800 \pm 0.00031$	0.00793	0.84
$\chi(4, 4) = 0.00635 \pm 0.00045$	0.00649	-2.15
$\chi(4, 5) = 0.00550 \pm 0.00060$	0.00580	-5.25
$\chi(4, 6) = 0.00495 \pm 0.00076$	0.00543	-8.79
$\chi(4, 7) = 0.00461 \pm 0.00094$	0.00520	-11.40
$\chi(4, 8) = 0.00435 \pm 0.00112$	0.00505	-13.96
$\chi(5, 2) = 0.01188 \pm 0.00027$	0.01161	2.31
$\chi(5, 3) = 0.00760 \pm 0.00044$	0.00738	3.06
$\chi(5, 4) = 0.00590 \pm 0.00063$	0.00580	1.69
$\chi(5, 5) = 0.00498 \pm 0.00083$	0.00503	-0.98
$\chi(5, 6) = 0.00444 \pm 0.00104$	0.00459	-3.35
$\chi(5, 7) = 0.00408 \pm 0.00128$	0.00432	-5.44
$\chi(5, 8) = 0.00386 \pm 0.00152$	0.00414	-6.87
$\chi(6, 2) = 0.01177 \pm 0.00036$	0.01142	3.01
$\chi(6, 3) = 0.00740 \pm 0.00057$	0.00708	4.43
$\chi(6, 4) = 0.00567 \pm 0.00081$	0.00543	4.42
$\chi(6, 5) = 0.00473 \pm 0.00106$	0.00459	3.11
$\chi(6, 6) = 0.00416 \pm 0.00133$	0.00411	1.28
$\chi(6, 7) = 0.00379 \pm 0.00162$	0.00380	-0.42
$\chi(6, 8) = 0.00358 \pm 0.00191$	0.00360	-0.65
$\chi(7, 2) = 0.01174 \pm 0.00046$	0.01131	3.80
$\chi(7, 3) = 0.00729 \pm 0.00072$	0.00691	5.48
$\chi(7, 4) = 0.00552 \pm 0.00101$	0.00520	6.09
$\chi(7, 5) = 0.00462 \pm 0.00132$	0.00432	7.03
$\chi(7, 6) = 0.00403 \pm 0.00165$	0.00380	6.01
$\chi(7, 7) = 0.00356 \pm 0.00199$	0.00347	2.43
$\chi(7, 8) = 0.00339 \pm 0.00234$	0.00325	4.36
$\chi(8, 2) = 0.01169 \pm 0.00056$	0.01124	3.96
$\chi(8, 3) = 0.00716 \pm 0.00088$	0.00680	5.30
$\chi(8, 4) = 0.00531 \pm 0.00122$	0.00505	5.23
$\chi(8, 5) = 0.00436 \pm 0.00160$	0.00414	5.30
$\chi(8, 6) = 0.00381 \pm 0.00201$	0.00360	5.91
$\chi(8, 7) = 0.00333 \pm 0.00242$	0.00325	2.51
$\chi(8, 8) = 0.00306 \pm 0.00284$	0.00301	1.62

MONTE CARLO CREUTZ RATIOS FROM WILSON'S METHOD

Bhanot and Creutz[8] and Ambjørn et al.[9] have reported data on Wilson loops for $U(l)_3$ from runs on a 16^3 lattice. Both groups fitted square Wilson loops to curves of the form

$$W(La, La) = \exp[-\sigma(\beta)L^2 + P(\beta)L + B(\beta)] \tag{21}$$

and reported values of the string tension $\sigma(\beta)$ for various β between 1 and 3.5. They did not report Creutz ratios or the values of any nonsquare Wilson loops. In order to extract Creutz ratios $\chi(L, L)$ for square loops from their data, we make the reasonable assumption that the Wilson loops $W[La, (L-1)$ and $W[(L-1)a, La]$, had they measured them, would have fitted the same curves as their $W(La, La)$s with L^2 replaced by $L(L-1)$ and L replaced by $L-\frac{1}{2}$. This assumption is implicit in their extraction of string tensions from those curves.

From this assumption it follows that the Creutz ratios $\chi(L, L)$, had they measured them, would have been given by the relation $\chi(L, L) \approx \sigma(\beta)$ with L taken somewhat beyond the midpoint of the range of Ls they used to fit their loops. Since Bhanot and Creutz fitted Wilson loops with values of L from 1 through 4, we identify in Table IV and Fig. 1 their $\sigma(\beta)$s with

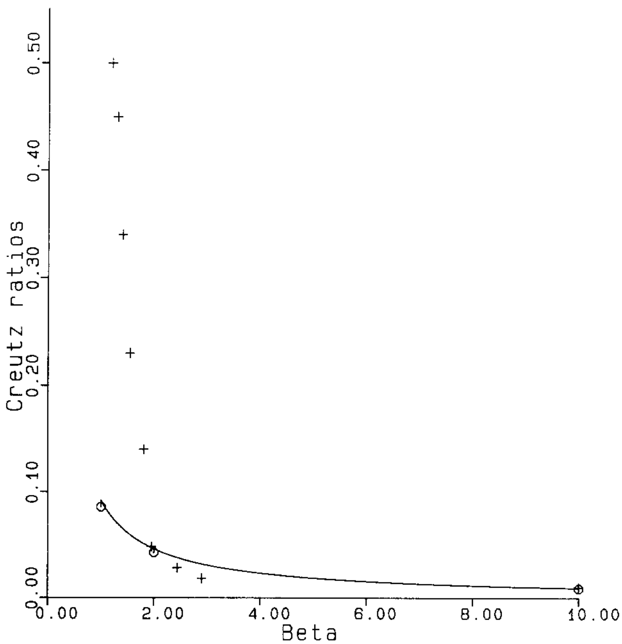

Fig. 1. Creutz ratios $\chi(3, 3)$ obtained from the simplicial method (octagons) and Wilson's method (pluses). The exact $\chi(3, 3)$s are represented by the solid curve.

Table IV. Creutz Ratios (from Ref. 8)

β	$\chi(3, 3)$	Exact χs	% error
1.2	0.5	0.07631	560
1.3	0.45	0.07044	540
1.4	0.34	0.06541	420
1.54	0.23	0.05946	290
1.8	0.14	0.05087	180
1.95	0.048	0.04696	2
2.44	0.028	0.03753	–25
2.9	0.018	0.03158	–43

$\chi(3, 3)$s. Since Ambjørn et al. fitted loops with values of L from 4 through 8, we identify in Table V and Fig. 2 their $\sigma(\beta)$s with $\chi(7, 7)$s. Of the 16 χs so inferred, 2 are within 6% of the exact values; 5 differ by 25 to 59%; and 9 are off by 150 to 1500%. The errors of these last 9 points cannot be ascribed to the crudity of our identification of χs with σs. Nor are they a

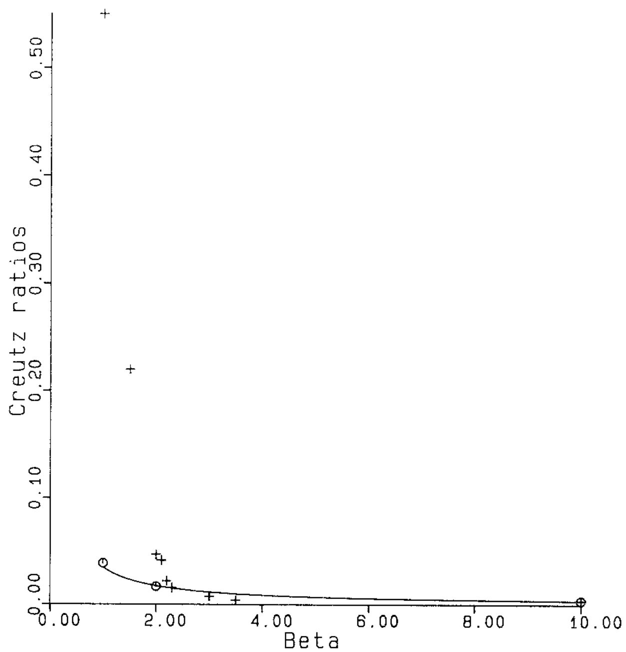

Fig. 2. Creutz ratios $\chi(7, 7)$ obtained from the simplicial method (octagons) and Wilson's method (pluses). The exact $\chi(7, 7)$s are represented by the solid curve.

Table V. Creutz Ratios (from Ref. 9)

β	$\chi(7, 7)$	Exact χs	% error
1.0	0.55	0.03471	1,500
1.5	0.22	0.02314	860
2.0	0.047	0.01736	170
2.1	0.041	0.01653	150
2.2	0.022	0.01578	39
2.3	0.016	0.01509	6
3.0	0.0075	0.01157	−36
3.5	0.0041	0.00991	−59

finite-size effect; in $U(1)_3$ with the Wilson action such effects are small on a 16^3 lattice,[9,16] because of the mass gap. These errors reflect the incorrect β dependence of their Wilson loops.

WILSON'S METHOD AT WEAK COUPLING

In order to examine the accuracy of Wilson's lattice gauge theory for weak coupling, we turn to the weak-coupling expansion of Müller and Rühl.[10] For an infinite lattice, the first two terms of their weak-coupling expansion for a $U(1)_3$ Wilson loop are

$$\ln[W(La, Ja)] \approx -\left[\frac{1}{\beta}+\frac{1}{3\beta^2}\right] W_1(La, Ja) \qquad (22)$$

By using the numerical values for $W_1(La, Ja)$ of Ambjørn et al.,[9] we find that the Wilson method on an infinite lattice should give the following Creutz ratios for $\beta = 10$: $\chi(3, 3) \approx 0.0099$, $\chi(4, 4) \approx 0.0069$, $\chi(5, 5) \approx 0.0053$, $\chi(6, 6) \approx 0.0043$, $\chi(7, 7) \approx 0.0036$, and $\chi(8, 8) \approx 0.0031$. These values differ from the exact χs by 8.6, 6.3, 5.0, 4.4, 4.8, and 3.1%, respectively, errors which are only about twice those of the simplicial method. Wilson's lattice gauge theory is thus far better at weak coupling than at intermediate or strong coupling. We used these weak-coupling estimates of $\chi(3, 3)$ and of $\chi(7, 7)$ in the figures to characterize the accuracy of Wilson's method at $\beta = 10$.

CONCLUSION

On the basis of the data reported here and displayed in the tables and figures, we conclude that the simplicial method is more accurate than

Wilson's lattice gauge theory for $U(1)_3$ at intermediate and strong coupling. The better accuracy of the simplicial method is due to its use of the action and domain of integration of the exact theory, unaltered apart from the granularity of the simplicial lattice. Wilson's action and domain of integration are close to those of the exact theory for weak coupling, but not for intermediate or strong coupling.

The action of the simplicial method possesses gauge invariance only up to the granularity of the lattice. For $U(1)_3$ in the temporal gauge, this lack of exact gauge invariance was unimportant; whether it would matter for a non-Abelian gauge theory is an open question. We are now investigating this question for $SU(2)_3$.

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